Problem set 3
Due date: 11th Feb

Part A (submit any three)

Exercise 19. (1) Show that the family of Normal distributions $\{N(\mu,\sigma^2) : \mu \in \mathbb{R} \text{ and } \sigma^2 > 0\}$ is not tight.
(2) For what $A \subset \mathbb{R}$ and $B \subset (0,\infty)$ is the restricted family $\{N(\mu,\sigma^2) : \mu \in A \text{ and } \sigma^2 \in B\}$ tight?

Exercise 20. Given a Borel p.m. μ on $\mathbb{R}$, show that it can be written as a convex combination $\alpha\mu_1 + (1-\alpha)\mu_2$ with $\alpha \in [0,1]$, where μ_1 is a purely atomic Borel p.m and μ_2 is a Borel p.m with no atoms.

Exercise 21 (Bernoulli convolutions). For any $\lambda > 1$, define $X_\lambda : [0,1] \to \mathbb{R}$ by $X(\omega) = \sum_{k=1}^{\infty} \lambda^{-k} X_k(\omega)$. Check that X_λ is measurable, and define $\mu_\lambda = \mathbf{m}X_\lambda^{-1}$. Show that for any $\lambda > 2$, show that μ_λ is singular w.r.t. Lebesgue measure.

Exercise 22. For $p = 1, 2, \infty$, check that $\|X - Y\|_p$ is a metric on the space $L^p := \{[X] : \|X\|_p < \infty\}$ (here $[X]$ denotes the equivalence class of X under the above equivalence relation).

Exercise 23. (1) Give an example of a sequence of r.v.s X_n such that $\liminf \mathbf{E}[X_n] < \mathbf{E}[\liminf X_n]$.
(2) Give an example of a sequence of r.v.s X_n such that $X_n \stackrel{a.s.}{\to} X$, $\mathbf{E}[X_n] = 1$, but $\mathbf{E}[X] = 0$.
(3) Suppose $X_n \geq 0$ and $X_n \to X$ a.s. If $\mathbf{E}[X_n] \to \mathbf{E}[X]$, show that $\mathbf{E}[|X_n - X|] \to 0$.

Part B (submit any two)

Exercise 24 (Alternate construction of Cantor measure). Let $K_1 = [0,1/3] \cup [2/3,1]$, $K_2 = [0,1/9] \cup [2/9,3/9] \cup [6/9,7/9] \cup [8/9,1]$, etc., be the decreasing sequence of compact sets whose intersection is K. Observe that K_n is a union of 2^n intervals each of length 3^{-n}. Let μ_n be the p.m. which is the "renormalized Lebesgue measure" on K_n. That is, $\mu_n(A) := 3^n 2^{-n} \mathbf{m}(A \cap K_n)$ for $A \in \mathcal{B}_{\mathbb{R}}$. Then each μ_n is a Borel p.m. Show that $\mu_n \stackrel{d}{\to} \mu$, the Cantor measure (which was defined differently in class).

Exercise 25 (A quantitative characterization of absolute continuity). Suppose $\mu \ll \nu$. Then, show that given any $\varepsilon > 0$, there exists $\delta > 0$ such that $\nu(A) < \delta$ implies $\mu(A) < \varepsilon$. (The converse statement is obvious but worth noticing). [**Hint**: Argue by contradiction].

Exercise 26. (1) Suppose $f : [a,b] \to \mathbb{R}$ is a Borel measurable function. Then, show that $g(x) := \int_0^x f(u)du$ is a continuous function on $[0,1]$.
(2) Let $f : [a,b] \times R \to \mathbb{R}$ be a bounded function. Assume that $f(x,\theta)$ is Borel measurable in the x for each fixed value of the θ. Assume also that f is continuously differentiable in θ for each fixed value of x. Further assume that $f(x,\theta)$ as well as $\frac{\partial f}{\partial \theta}(x,\theta)$ are uniformly bounded in (x,θ). Then, justify the following "differentiation under integral sign"

$$\frac{d}{d\theta}\int_a^b f(x,\theta)dx = \int_a^b \frac{\partial f}{\partial \theta}(x,\theta)\, dx$$

Exercise 27 (Moment matrices). Let $\mu \in \mathcal{P}(\mathbb{R})$ and let $\alpha_k = \int x^k d\mu(x)$ (assume that all moments exist). Then, for any $n \geq 1$, show that the matrix $(\alpha_{i+j})_{0 \leq i,j \leq n}$ is non-negative definite. [**Suggestion:** First solve $n = 1$].